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NON-TRADITIONAL "TANGENT" AND "COTANGENT" VECTORS OF ROTATION IN ORIENTATION AND NAVIGATION PROBLEMS

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Proposed for use in problems of orientation, navigation and Euler rigid body dynamics (RB) conjugate ("tangent" and "cotangent") vectors of finite Euler rotation. Modules of these vectors are proportional to tangent and cotangent of a quarter angle of rotation. Contrary to the classical Rodrigues' vector, they do not become infinite within the complete RB rotation. Kinematic differential equations for these vectors have common solution (in the Cauchy's form). New dynamic polar differential equations are obtained from the classical dynamic Euler-Poisson equations as a result of replacement of angular velocity vector by the vector angular momentum, as well as replacement of three directional Poisson cosines by the conjugate vectors coordinates. In the case of Euler, three obtained scalar dynamic differential equations have two classical first integrals (of energy and areas). Three new dynamic equations are considered as integrable ones. These equations explicitly determine the whole RB orientation, not just its vertical, like three Poisson equations do. Conjugate vectors $\vec{\tau}$, $\vec{\rho}$ *are considering as parts of vectors with non-traditional non-normalized (non-Hamiltonian) rotation quaternion and biquaternion. Such non-normalized quaternions can be effectively used instead of classical normalized quaternions and biquaternions (with parameters of Rodrigues-Hamilton) in different «particular» tasks of inertial orientation and navigation, including RB orientation control problem.*

1. In inertial orientation and navigation are of particular interest dimensional rotation vectors having a minimum number of independent parameters, corresponding to three degrees of freedom of the RB with a fixed point [1]. In classical Euler's problem of RB dynamics with a fixed point [1], the differential Euler-Poisson equations are commonly used. Three Poisson equations determine only the earth vertical reference in relation to RB, not the whole RB orientation. Moreover, it is presumed possible to determine RB orientation through the Euler angles with additional integration of expression for time derivative of RB precession angle [1, page 16]. But the direct integration of only three Poisson equations basically does not solve the problem of determining RB orientation in three-dimensional space. In order to explicitly determine RB orientation, it is necessary to have at least six directional cosines. However, in this case, one must integrate nine differential equations instead of six Euler-Poisson equations. In this regard, the possibility of using well-known [2-10] three-dimensional rotation vectors in RB dynamics is appealing, because they have three independent coordinates.

2. Classical Rodrigues's [2,3] vector has a module with tangent $\tan(\varphi/2)$ of half-angle φ of finite rotation of RB. This vector can not be used in the condition of $\varphi = \pi$ in the problems of RB dimensional orientation. Conjugate rotation vectors $\vec{\tau} = \tau\vec{k}$, $\vec{\rho} = \rho\vec{k}$, where $\tau = k_\tau \tan(\varphi/4)$, $\rho = k_\rho \cot(\varphi/4)$ (k_τ , k_ρ are arbitrary constant coefficients), $\vec{k}$ – unit vector of Euler axis of RB finite rotation are of the most interest (by $0 < \varphi < 2\pi$) because of its conjugate features [3].

The article deals with the *conjugative polar* non-linear vector kinematic differential equations of RB rotation in the transformation of lineal vector $\vec{\omega}(t)$ of RB angular velocity [3,7]:

$$\vec{\tau}^{\,*} = \tau'\theta_{(\tau)}\vec{\omega}, \; \vec{\rho}^{\,*} = \rho'\theta^{T}_{(\rho)}\vec{\omega}, \qquad (1)$$

where $\vec{\tau}^{\,*} = d\vec{\tau}/dt$, $\vec{\rho}^{\,*} = d\vec{\rho}/dt$ – local derivative of vector in the time t (derivative towards some connected with RB coordinate base); $\tau' = \partial\tau/\partial\varphi$, $\rho' = \partial\rho/\partial\varphi$ – partial derivatives of modules τ, ρ of vector of angle φ, defined as function $\tau' = (k_\tau + \tau^2/k_\tau)/4$, $\rho' = -(k_\rho + \rho^2/k_\rho)/4$; $\theta_{(\tau)}$, $\theta^{T}_{(\rho)}$ – orthogonal operators:

$$\theta_{(\tau)} = \mathbf{E} + 2\left(k_\tau\mathbf{T} + \mathbf{T}^2\right)/\left(k_\tau^2 + \tau^2\right),$$

$$\theta^{T}_{(\rho)} = \mathbf{E} + 2\left(-k_\rho\mathbf{R} + \mathbf{R}^2\right)/\left(k_\rho^2 + \rho^2\right).$$

Therein $\mathbf{T}$, $\mathbf{R}$ are skew-symmetric operators of vector multiplication [3,5,8], satisfies the identities $\mathbf{T}\vec{\tau} = \vec{\tau}\times\vec{\tau} = \vec{0}$, $\mathbf{T}\vec{\rho} = \vec{\rho}\times\vec{\rho} = \vec{0}$, $\vec{0}$ – zero vector, $\mathbf{E}$ – unit operator. Operators $\theta_{(\tau)}$, $\theta_{(\rho)}$ satisfy the identity

$\left(\theta_{(\tau)}\theta_{(\rho)}\right)^2 = \mathbf{E}$, determined conjugacy [3] of vectors $\vec{\tau}$, $\vec{\rho}$ and conjugate equations as features of “duality” and isomorphic correspondence.

3. Conjugate equations have the first («trigonometric») integral in the form of scalar product $(\vec{\tau} \cdot \vec{\rho}) = \tau\rho = k_\tau k_\rho = C$ (arbitrary constant) and admit simple and graphic kinematic interpretation – polar wobbling-nutational model [3,9] with arbitrary vector $\vec{\omega}(t)$ and for any RB. In the first equation vector $\vec{\omega}$ is transformed (at every moment of the travelling time of RB) by operator $\theta_{(\tau)}$ into vector $\vec{\omega}_\tau$ as a result of precession в $\vec{\omega}$ on the angle $\psi_{\theta\tau} = \varphi/2$ (rotation vector $\vec{\omega}$ with module $\omega(t)$ about the surface of some circular cone of «velocity» precession cone angle $2\upsilon_\omega$) around Euler axis with unit vector $\vec{k}$. Angle υ_ω – nutation angle (vector deviation $\vec{\omega}$ from the unit vector $\vec{k}$). And then a vector $\vec{\omega}_\tau$ is multiplied by scalar operator $\tau' E$. At the second equation vector $\vec{\omega}$ precesses (rotates n the surface of that) at the angle $\psi_{\theta\rho} = (\pi + \varphi/2)$ and is multiplied by scalar operator $\rho' E$. Angle υ_ω of nutation is determined from scalar product $(\vec{\omega} \cdot \vec{\tau})$, $(\vec{\tau}^* \cdot \vec{\tau})$ or $(\vec{\omega} \cdot \vec{\rho})$, $(\vec{\rho}^* \cdot \vec{\rho})$.

On the basis of such interpretation – equation (1) models come out, for example, kinematic nonholonomic equality:

$$\left((\vec{a} \cdot \vec{b})^2 + ((\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}))\right) \Big/ \left((\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})\right) = \sin^2 \frac{\varphi}{2} + \cos^2 \frac{\varphi}{2} = 1, \tag{2}$$

where $\vec{a} = \vec{\omega} \times \vec{\tau}$, $\vec{b} = \left(\theta_{(\tau)}\vec{\omega}\right) \times \vec{\tau}$.

The equality with arbitrary constant $C_\tau = k_\tau^2$, comes out from equality, for example:

$$C_\tau = \tau^2 \frac{(\vec{a} \cdot \vec{a}) + (\vec{a} \cdot \vec{b})}{(\vec{a} \cdot \vec{a}) - (\vec{a} \cdot \vec{b})}. \tag{3}$$

Equalities, analogous to that equalities, comes out of vector change $\vec{\tau}$ to vector $\vec{\rho}$.

4. Under the numerical integration [2-4] of conjugate equations such values are to exist $k_{\tau*}$ and $k_{\rho*}$ of coefficients k_τ, k_ρ, where equality is provided $\tau' = 1$ и $\rho' = 1$. Then, for example, $k_{\tau*} = 2 + \sqrt{4 - \tau^2}$, $k_{\rho*} = -2 - \sqrt{4 - \rho^2}$, ($\tau \le 2$, $\rho \le 2$) vector transformation $\vec{\omega}$ into derivatives $\vec{\tau}^*$, $\vec{\rho}^*$ will be exercised by orthogonal operators $\theta_{\tau*}$, $\theta_{\rho*}^T$, where at every step of calculation new values of “adaptive” coefficients $k_{\tau*}$, $k_{\rho*}$ (instead of constant coefficients k_τ, k_ρ) are used. But in fact integrate conjugate equations in the form of orthogonal transformation $\vec{\tau}_*^* = \theta_{\tau*}\vec{\omega}$, $\vec{\rho}_*^* = \theta_{\rho*}\vec{\omega}$ when known (approximately «measured» by integrating gyroscopes) of kinematic integrals (vector modules $\vec{\tau}^*$, $\vec{\rho}^*$) $\tau_* \approx \rho_* \approx q + c_q \approx \varphi$ from scalar functions $d\tau_*/dt = \overset{*}{\tau}_*(t)$, $\overset{*}{\rho}_*(t)$, $\omega(t)$ (c_q – arbitrary constant, $\omega(t)$ – module of vector).

5. Conjugate equations have general solutions in the Cauchy form [3,6]. General solution of the first equation in (1) determines the form of group operations of multiplication and division in group of Lie vectors $\vec{\tau}$ [3,7]. In this group inverse element $\vec{\tau}^{-1}$ is equal to adverse vector, i. e. $\vec{\tau}^{-1} = -\vec{\tau}$, but unit element is equal to zero vector $\vec{0}$.

6. New polar dynamical differential equations of RB rotation are obtained on the basis of kinematic conjugate equations.

In the classical Euler case, for example, the problem of integration of the system of six dynamical differential equations of Euler-Poisson [1] reduces [7,9] to integration of the system of three dynamical equations with two independent classical first integrals (the energy and area integrals). These two integrals are enough to consider the system of three equations to be integrated [1].

In this case change is introduced in the conjugate equations $\vec{\omega} = \mathbf{S}^{-1}\vec{g}$, where $\vec{g}$ – constant vector of angular momentum (constant module and in the line of supporting basis $\mathbf{I}$); $\mathbf{S}^{-1}$ – inverse operator S of RB inertia (constant in connected with RB basis $\mathbf{J}$).

Contained of conjugate equations, for example, polar matrix differential equation with vector coordinates $\vec{\tau}$ has the form [9]:

$$\vec{\tau}_J^* = \tau'\left(\theta_{(\tau)J} S_J^{-1} \theta_{(\tau)J}^T\right) \theta_{(\tau)J}^T \vec{g}_I , \qquad (4)$$

where $\vec{\tau}_J^* = \left[\dot{\tau}_{j1} \quad \dot{\tau}_{j2} \quad \dot{\tau}_{j3}\right]^T$ – column matrix with derivative coordinates $\vec{\tau}^*$ in basis $\mathbf{J}$; $\theta_{(\tau)J}$ – matrix (3×3) of operator $\theta_{(\tau)}$ in basis $\mathbf{J}$; S_J – diagonal matrix (3×3) with three constant the main moments of RB inertia; S_J^{-1} – inverse matrix, $g_I = \left[g_{i1} \quad g_{i2} \quad g_{i3}\right]^T$ – constant column matrix with vector coordinates $\vec{g}$ in basis $\mathbf{I}$.

Three vector coordinates $\vec{\tau}$ definitely determine RB orientation (in contrast to three direction cosines determined orientation of just one unit vector of supporting basis $\mathbf{I}$ in Poisson equations). In the problem of synthesis of RB orientation control law (determination of control moments) applying functions of Lyapunov equation is used together with classical Euler's dynamical equations.

7. Conjugate vectors $\vec{\tau}$, $\vec{\rho}$ are considering as parts of vectors with non-traditional [10, 11] non-normalized (non-Hamiltonian) rotation quaternion and biquaternion. Such non-normalized quaternions can be effectively used instead of classical normalized quaternions and biquaternions [2, 10] (with parameters of Rodrigues-Hamilton) in different «particular» tasks of inertial orientation and navigation, including RB orientation control problem.

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